

# Case Study: Visualization and Analysis of High Rayleigh number — 3D convection in the Earth’s Mantle

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## Abstract

Data sets from large-scale simulations (up to  $501^3$  grid points) of mantle convection are analyzed with volume rendering of the temperature field and a new critical point analysis of the velocity field. As the Rayleigh number  $Ra$  is increased the thermal field develops increasingly thin plume-like structures along which heat is convected. These eventually break down and become turbulent. Visualization methods are used to distinguish between various models of heat conductivity and to develop an intuitive understanding of the structure of the flow.

**Keywords:** Mantle convection, plumes, volume rendering, unsteady flow, feature extraction, critical points

## 1 INTRODUCTION

Mantle convection is a problem of considerable interest in geophysical fluid dynamics, because of the richness in the physics such as variable properties, in particular the recent work which focusses on variable thermal conductivity [1]. Since the late 1980’s mantle convection has been modelled in 3-D but at relatively low Rayleigh numbers like  $10^5$  [2]. The resolution used at that time was relatively low about 2 million grid points. Recently we have conducted some high Rayleigh number calculations with close to  $10^8$  grid points [3]. Variable thermal conductivity introduces new nonlinearities into the temperature equation, making it looking like the Burgers’s equation near the boundary layers, because of the presence of the square of the temperature gradient, which dominates over the Laplacian of the temperature. It is more difficult to solve numerically and requires higher spatial resolution than for constant thermal conductivity. The spatial structures of plumes associated with variable thermal conductivity are different from those of constant conductivity [4]. In this case study we extend our previous work on high Rayleigh number convection [5] to even higher Rayleigh numbers on the order of  $10^9$  and improve on both the visualization tools and techniques used in our previous attempts to visualize mantle convection processes [6].

## 2 MATHEMATICAL MODEL

Mantle convection can be modelled as a fluid at infinite Prandtl number subject to the Boussinesq approximation. The motion of this fluid is solved for in a 3-D Cartesian geometry, with the vertical axis ( $z$ ) pointing downward (Figure 1). The equations of motion are the temperature equation with variable thermal conductivity,

$$\frac{DT}{Dt} = \nabla(\kappa(T, p)\nabla T)$$

and the momentum equation (which does not depend on the variable viscosity or conductivity)

$$\nabla^2 \mathbf{u} - \nabla p - Ra T e_z = 0,$$

$\frac{DT}{Dt}$  is the substantive derivative of temperature,  $T$  is the deviation of temperature from a constant reference state,  $\mathbf{u}$  is the velocity vector,  $p$  is the pressure in excess of the hydrostatic value,  $T$  is the temperature deviation from the background state,  $Ra$  is the Rayleigh number and  $e_z$  is the unit vector along the  $z$  direction which is aligned with the gravity vector. The Rayleigh number  $Ra = \alpha g \Delta T d^3 / (\nu \kappa)$  where  $\alpha$  is the thermal expansivity,  $g$  is the gravitational acceleration,  $\Delta T$  is the temperature drop across the mantle,  $d$  is the depth of the mantle,  $\nu$  is the kinematic viscosity and  $\kappa$  is the surface value of the thermal diffusivity. As a result of variable conductivity, the terms on the right hand side of the temperature equation are non-linear functions of  $T$  as opposed to the simple linear diffusion term in the constant conductivity case. Boundary conditions on temperature are  $T = 0$  at  $z = 0$  (cold top) and  $T = 1$  at  $z = 1$  (heated bottom). The dimensional temperature at the Earth’s core-mantle boundary is uncertain and may range between 3200 K to 4500 K [7]. The physical domain has an aspect-ratio of  $4 \times 4 \times 1$  for modelling 3-D mantle convection in a cartesian geometry. This geometric configuration has been used the past [8] to model dynamics of mantle convection at lower Rayleigh number than the ones considered here. The thermal mantle conductivity has two components, the radiative conductivity and the lattice conductivity. These expressions, given in [9, 10] are based on experimental phonon lifetimes and reflectance data.

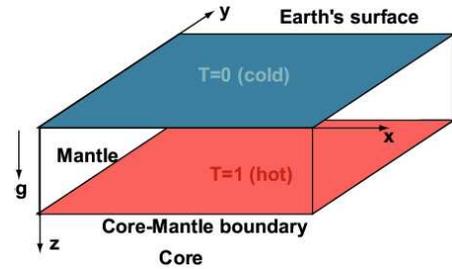


Figure 1: Geometry of mantle convection

## 3 NUMERICAL METHOD

The flow is assumed to be periodic in the  $x - y$  (horizontal) directions. The grid is uniform in all three directions. We have used an Alternating-Direction-Implicit scheme together with a finite differ-

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ence method to solve the variable conductivity temperature equation. A Smolarkiewicz/finite-difference scheme is used for the constant conductivity case. The momentum equation is solved in the spectral domain using Fast Fourier Transform along the horizontal direction and second-order finite-differences in  $z$ . Free-slip boundary conditions are imposed on the horizontal surfaces, while the vertical boundaries are impermeable.

Data from several simulations are used for this study. A low resolution  $97^3$  simulation at  $Ra = 10^6$  was chosen to illustrate the time-dependent nature of the data. These simulations (for constant and variable conductivity) were run for 8000 time steps, from which data at 300 time steps was stored and analyzed to study the time evolution of structural information. Two very high resolution simulations at  $Ra = 10^8$  and  $Ra = 10^9$  were conducted at resolutions of  $401^3$  and  $501^3$  grid points respectively. This data was generated for constant conductivity.

## 4 AMIRA

We have chosen the visualization package Amira [11] as the tool of choice for the display of complex flows. Amira, similarly to AVS, Iris Explorer, and Data Explorer, allows the user to construct flowcharts interactively from a collection of predefined modules. Although the three aforementioned products have longevity and are quite capable of representing a wide variety of flow configurations, none can do so with the efficiency afforded by Amira. Amira is built on top of Inventor, a well-known SGI product, strongly object-oriented, and itself based on OpenGL. While OpenInventor provides facilities for high-level interaction with the data through embedded widgets and the means to attach virtually any variable to an action, Amira interacts with the data at a higher level. For example, it can manipulate structured and unstructured meshes, slice through grids at interactive speeds, etc. There are sophisticated isosurface, line integral convolution, illuminated streamlines, and volumetric tools that harness the latest advances in texture-based visualization algorithms. As a result, Amira can manipulate extremely large data sets very efficiently. For example, it takes under 30 seconds to render a  $400^3$  data set on a Precision 530 Workstation (1.7 GHz Xeon chip, 1 Gbyte memory, 400 MHz bus). All the results in this paper were generated on this workstation.

Amira is fully extensible through a well-defined object-oriented hierarchy of data structures that define grids and data types. Wrappers around many Inventor features are also provided. The algorithms discussed below have been implemented as Amira modules written by one of the authors. They are imported into Amira and appear as any other Amira module in the visual flowcharts. One feature of Amira that is particularly useful is its use of Tcl as a scripting language, which can also be used to construct modules and enhance their functionality.

## 5 THERMAL PLUMES

One of the fundamental challenges when analyzing flow turbulence is the identification of the so-called coherent structures. These have been defined in many ways, and to date there is still no real consensus. However, one definition that has appeal is to consider a coherent structure as a localized region of the flow that remains correlated over spatial and temporal scales much larger than the scales of the random flow component. Examples of such structures include turbulent spots in wall-bounded flow undergoing transition, hairpin vortices in wall-bounded turbulent flow, and elongated dissipative vortices in isotropic turbulence [12]. Such structures have been studied extensively, but aside from vortices, there have been very few algorithms developed to extract them. Feature extraction

is seen as a necessary component required to successfully handle very large data sets in complex fluids.

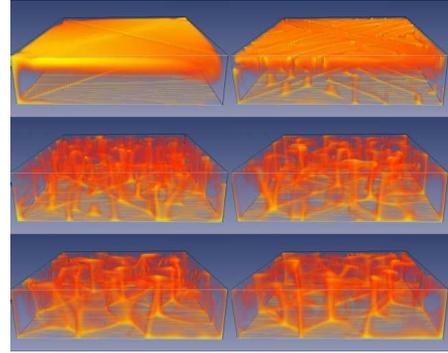


Figure 2: Plumes in  $97^3$  constant conductivity.

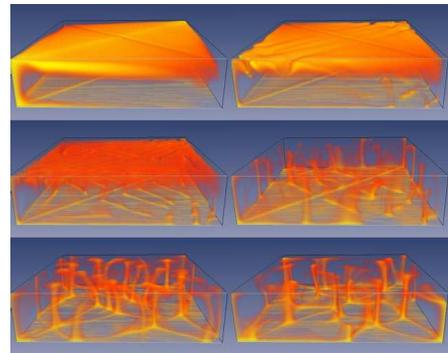


Figure 3: Plumes in  $97^3$  variable conductivity

When  $Ra$  is sufficiently high, the temperature field develops turbulent characteristics, reminiscent of the turbulent velocity fields found at high Reynolds number. The coherent structures associated with the temperature field are called thermal plumes. They exist at lower Rayleigh numbers, but develop a range of spatial scales as  $Ra$  is increased. Plumes play an important role in the dynamics of mantle convection because they determine the style of mantle convection and the global heat transfer. Until recently, the conductivity was assumed to be constant. The visualizations shown herein demonstrate that variable conductivity can change the spatial distribution and the strength of the plumes, thus affecting the dynamical processes through which heat is conducted through the mantle.

Figures 2 and 3 are six frames taken from an animation of a volumetric representation of the temperature field at  $Ra = 10^6$ . The color map was chosen to emphasize the thermal plumes. The first two images are at early time, the last two at times when the flow has become quasi-stationary. Comparing the constant and variable conductivity cases, we note that when the conductivity is constant, the plumes develop faster and their spatial density is higher. In both cases, the structure of the plumes is approximately constant: their height and shape don't vary much. On the heated plate (bottom), the temperature develops thin structures from which the plumes emanate. A better understanding of these structure is a focus of current research. Qualitatively, the thermal plumes are tube-like; they are primarily vertical and extend from the bottom to the top plate.

As  $Ra$  is increased further, the plumes become thinner and more irregular. When  $Ra$  reaches the  $10^7$  range, the temperature field becomes turbulent [5]. Figures 4 and 5 show the temperature field at a stage when the flow is quasi-stationary. The plumes have become more irregular, particularly at  $Ra = 10^9$ . Many plumes no longer

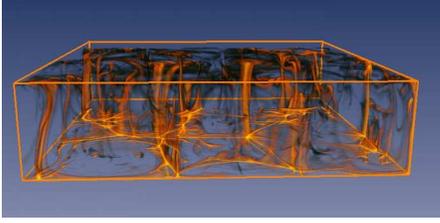


Figure 4:  $401^3$  temperature field at  $Ra = 10^8$ . Constant conductivity.

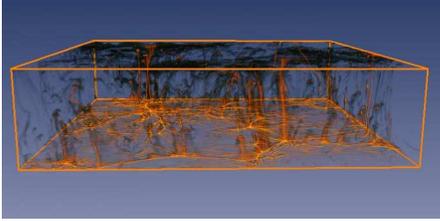


Figure 5:  $501^3$  temperature field at  $Ra = 10^9$ . Constant conductivity.

extend throughout the mantle. To get a better idea of the structure of the plume,  $x-z$  and  $x-y$  cross-sections are presented in Figures 6 and 7 respectively. The color maps correspond to those used in the volume-rendered images, including transparency. This was done to reduce accidental misinterpretation of the images when comparing plume slices and their 3-D representations. The top frame of each figure corresponds to  $Ra = 10^8$ , the bottom frame to  $Ra = 10^9$ . From the  $x-z$  profiles, it is immediately apparent that the plumes are thinner at higher  $Ra$ . However, the thicker plumes are not immediately seen from the  $x-y$  cuts. What can be deduced through these visualizations is that the plumes originate from the heated plate at the intersections of the “canals” of local temperature maximum. The complex structure of the temperature field near the bottom plate at high  $Ra$  is directly linked to the higher complexity of the plume structure.

From these images, we deduce that plumes under high Rayleigh number cover a very small region of the 3-D volume, which decreases as even high  $Ra$ . Thus the problem of visualizing plumes becomes more difficult at higher  $Ra$ . This would have been difficult to ascertain without the aid of advanced visualization software. The next step is to develop techniques to extract these plumes and store them separately for analysis. We assume that a better understanding of their spatial distribution, their shape, and their volume would provide geoscientists with useful information to help their modelling efforts. Defining these plumes in a quantitatively is not clear. Although the eye can clearly identify them from the 3D images, one must remember that the eye sees temperature integrated along a ray. However, one clearly identifies the plumes from the

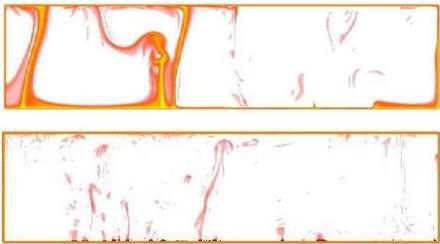


Figure 6: Typical  $x-z$  slice through the temperature field. Top:  $Ra = 10^8$ , bottom:  $Ra = 10^9$ . Constant conductivity.

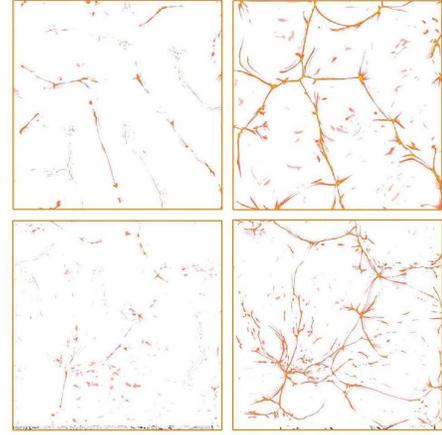


Figure 7: Typical  $x-y$  slice through the temperature field. Top:  $Ra = 10^8$ , bottom:  $Ra = 10^9$ . Constant conductivity.

cross sections. The temperature gradient will play an important role in identifying these plumes. Unfortunately, neither the temperature or the gradient is constant along the plume edge (there are strong and weak plumes). At this stage, the color maps used were generated by trial and error. A slight change in the map can easily make the plumes disappear. We have begun investigating the rendering of the temperature gradient field. Although the initial results are encouraging, we are seeking a more quantitative approach that would allow the color maps to be correctly built ab initio.

## 6 VELOCITY FIELD

What is the relationship between the structure of the thermal plumes and that of the velocity field? The temperature field clearly indicates upswelling and downswelling motion between the bottom and top plates, but what is the precise mechanism by which this occurs? The temperature can vary either through thermal conduction, or by convection by the velocity field. Thus it is of some interest to better understand the time-evolution of the velocity field.

In this paper, we consider a new use of critical points both as a means of flow representation in a highly compressed form and as a way to provide qualitative information about the flow. Critical points of the velocity field are intrinsically linked to the nonlinear dynamics of the flow [13]. In 3D flows, their structure provides clues regarding whether pathlines intersect planes along continuous curves or chaotic point distributions. In visualization, critical points have been used as a means of simplifying the representation of vector fields [14].

Critical points are points of a velocity field where the flow vanishes. The structure of the flow in a neighborhood of a critical point  $\mathbf{x}$  is easily derived from the linear approximation to the vector field about  $\mathbf{x}$ . The type of the critical point is determined by the eigenvalues of the Jacobian of the velocity field evaluated at  $\mathbf{x}$  is often used to characterize the nature of the flow (rotational, divergent, swirling, etc.). In three-dimensions, the interpretation of critical points is more difficult [15].

To date, critical points have not been exploited in time-dependent convective flows. The sparsity of critical points in 3-D flows leads us to an alternate technique that is meant to visualize the flow from the perspective of symmetry and global complexity. To accomplish this, we replace the 3-D velocity field  $\mathbf{u}(x, y, z) = (u, v, w)$  by three 2-D velocity fields  $\mathbf{u}_z(x, y) = (u, v)$ ,  $\mathbf{u}_x(y, z) = (v, w)$ , and  $\mathbf{u}_y(z, x) = (w, u)$ , restricted to the  $x-y$ ,  $x-z$  and  $y-z$  planes respectively. The subscripts  $x, y, z$  remind us that these 2-D fields are parameterized by the third dimension. Next, we compute

the critical points of  $\mathbf{u}_z(x, y)$  in all  $x - y$  slices and assign them a color. We do the same for the critical points in the  $x - z$  and  $y - z$  slices. Each critical point is displayed as a small cube. We pre-compute the critical points of the projected velocity fields that correspond to the 300 time slices of the  $97^3$  data set at  $Ra = 10^6$ . We have found approximately 500 and 5000 critical points in each frame.

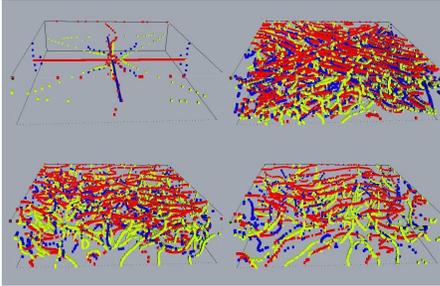


Figure 8: Critical points taken from 6 frames of an animation. Constant Conductivity.  $96^3$ .

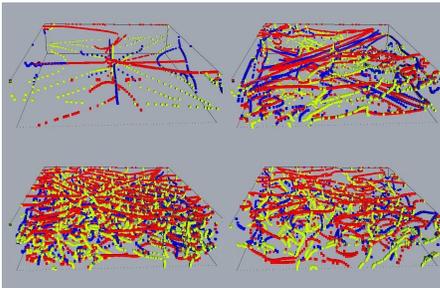


Figure 9: Critical points taken from 6 frames of an animation. Variable Conductivity.  $96^3$ .

The results for the constant and variable conductivity data are shown in Figures 8-9 respectively. The top-left image corresponds to an early stage in the simulation, before the formation of the plumes. In the top-right image, the complexity of the flow has clearly increased, as determined by the number and more apparently random distribution of the critical points. Note however, that the variable conductivity case maintains some structure for a longer time, consistent with the observations based on volume rendering of the temperature. The bottom-right image is representative of the state of the flow when it is quasi-stationary. We have constructed a module for Amira that can interactively browse through the critical points of all 300 sets data. The animation can be controlled by the software, or the user can interactively scan through the data for the purpose of exploration. Work is in progress to use this tool as a navigation device to zoom in on features of interest or to show alternate flow representations at the times selected by the user.

## 7 CONCLUSIONS

We have applied state-of-the-art visualization tools to some large-scale time-dependent mantle convection data sets to better understand the underlying structure of the temperature field, which becomes turbulent beyond a critical Rayleigh number. We have used the notion of critical points to construct a skeletal representation of the flow that helps characterize inherent symmetries and complexity. While volumetric rendering remains a fundamental tool to help understand the geometric structural intricacies of thermal plumes, there remains the task of developing robust algorithms for

automatic color map generation and plume extraction. This work is fundamental to research activities Earthscope [16].

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